§ S^3，SU(2)，SO(3)，Killing vector fields and spinning tops

Physics and Mathematics

Q is the group of unit quaternions。

$$SU(2)=\left\{\begin{bmatrix} a & -b* \\ b & a* \end{bmatrix} \middle| a,b\in C, |a|^2+|b|^2=1\right\}$$

1. $S^3 \cong SU(2)$ via Q。S^3具標準的 SU(2)左不變量。

$$S^3=\left\{q=a+bi+cj+dk \,\middle|\, a^2+b^2+c^2+d^2=1\right\}$$

2. SO(3) as a Quotient of S^3

- There is a **double covering map** from S^3 to SO(3), given by the action of unit quaternions on $\mathbb{R}^3$ via conjugation:

$$R_q(v)=qvq^{-1}$$

where v is treated as a **pure quaternion** ($v=xi+yj+zk$).

- The kernel of this map is $\{\pm 1\}$, meaning:

$$SO(3)\cong S^3/\{\pm 1\}$$

This shows that SO(3) is not simply connected，its findamental group is Z_2，means it has a double cover by S^3。

4. Physical and Mathematical Significance

- **Quantum Mechanics:** The **spin group** $\mathrm{Spin}(3)$ is isomorphic to S^3 and serves as the **double cover** of SO(3). This explains why **spin-½ particles require a 720° rotation to return to their original state**.
- **Computer Graphics & Robotics:** Rotations are often represented using **quaternions** (elements of S^3) to avoid gimbal lock and ensure smooth interpolation (SLERP).
- **Topology & Geometry:** SO(3) is a non-trivial quotient of S^3, making it an interesting example in fiber bundles:

$$S^3 \to SO(3) \to \mathbb{RP}^3$$

where SO(3) is homeomorphic to **real projective 3-space** ($\mathbb{RP}^3$).

3. $I\times S^2 \to S^3$

4. $X=-z\partial_y+y\partial_z, Y=-z\partial_x+x\partial_z, Z=-y\partial_x+x\partial_y$ are Killing fields generate Lie algebra so(3)。

The relationship between Killing vector fields and SO(3) arises in the study of symmetries of Riemannian manifolds，particularly in the context of rotational symmetries of 3-dimensional spaces。

3. SO(3) and Killing Vector Fields

- The isometry group of **flat** Euclidean space $\mathbb{R}^3$ includes SO(3) (rotations) as a subgroup.
- The generators of SO(3) correspond to **Killing vector fields** that represent infinitesimal rotations.
- The Lie algebra $\mathfrak{so}(3)$ is **spanned by three Killing vector fields**, which can be written in terms of angular momentum generators:

$$X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

These satisfy the Lie algebra comutation relations $[X_i, X_j] = \varepsilon_{ijk} X_k$ where ε_{ijk} is the Levi-Civita symbol

5. Spinning top SO(3)

1. SO(3) and Rigid Body Rotations

The **rotation group** SO(3) describes all possible orientations of a rigid body in three-dimensional space. Every **proper rotation** can be represented as an element of SO(3), which has three degrees of freedom (corresponding to three independent rotation axes).

A spinning top, as a rigid body with one fixed point, undergoes motion described entirely by elements of SO(3).

2. Configuration Space of a Spinning Top

The **configuration space** of a spinning top is given by the set of all possible orientations of the body relative to an inertial frame, which can be parametrized by an element of SO(3).

- Any orientation of the top can be described by a rotation matrix $R(t) \in \mathrm{SO}(3)$.
- The equations of motion describe how $R(t)$ evolves in time under the influence of external forces (like gravity) and internal angular momentum.

3. Equations of Motion inSO(3)

The motion of a spinning top is governed by Euler equation for a rigid body in its body frame：

$$I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1 \qquad I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = \tau_1$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \tau_3$$

Where

1. $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity in the body frame
2. I_1, I_2, I_3 are the principal moments of inertia
3. $\tau = (\tau_1, \tau_2, \tau_3)$ is the external torque (e.g. due to gravity)

4. When $\tau = 0$，the motion is purely free rotation

Since SO(3) is a **Lie group**, the time evolution of the rotation matrix $R(t)$ is given by:

$$\frac{dR}{dt} = R\Omega$$

where Ω is the **skew-symmetric angular velocity matrix**, defined as:

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

This structure shows that angular velocity lives in the **Lie algebra** $\mathfrak{so}(3)$, the tangent space to SO(3) at the identity.

4. SO(3) and Conservation Laws

- The spinning top exhibits **conserved quantities**, which are related to SO(3) symmetries:
  - **Total angular momentum** $L = I\omega$ is conserved in the absence of external forces.
  - **Energy** $E = \frac{1}{2}\omega^T I\omega$ is conserved.
  - If gravity acts, the projection of angular momentum along the vertical axis is conserved.

These conserved quantities arise due to **Noether's theorem**, since SO(3) symmetry leads to conservation of angular momentum.

5. Special Cases of Spinning Top Motion

(a) Free Rotation (Euler Top)

- If no external forces act ($\tau = 0$), the top undergoes **free rotation**.
- The motion is determined by **Euler's equations**, and the orientation evolves via SO(3) transformations.
- The angular momentum remains **fixed in the inertial frame**, but the top **precesses** in the body frame.

(b) Heavy Symmetric Top (Lagrange Top)

- When gravity is present, and the top is **axisymmetric** ($I_1 = I_2 \neq I_3$), the motion involves:
  - **Precession**: The axis of the top traces a cone.
  - **Nutations**: Oscillations in the tilt angle.

(c) Sleeping Top (Stable Rotation)

- If the top spins fast enough, it remains **upright** due to gyroscopic stability.
- This occurs when angular momentum along the symmetry axis is large.

6. SO(3) and the Euler Angles Representation

One way to describe the spinning top's orientation is using **Euler angles** (ϕ, θ, ψ), where:

- ϕ (precession): Rotation about the **vertical axis**.
- θ (nutation): Tilt of the top's symmetry axis.
- ψ (spin): Rotation around the top's own axis.

The rotation matrix R in SO(3) can be expressed in terms of Euler angles:

$$R = R_z(\phi)R_x(\theta)R_z(\psi)$$

where R_x, R_z are standard rotation matrices. The equations of motion for the angles describe the top's dynamics in terms of SO(3).

7. Lie Group and Hamiltonian Formulation
   Since SO(3) is a Lie group，its dynamics can also be formulated using Lie-Poisson equations or in the Hamitonian framwork，where
   (1) The phase space is the cotangent bundle $T^*SO(3)$，describing the evolution of angular momentum and orientation
   (2) The Poisson bracket struture is related to the Lie algebrs $so(3)^*$
   For integrable cases (like Kovalevskaya top)，the system has enough conserved quantites to be solved exactly。

Conclusion：

1. The motion of a spinning top is naturally decribed by SO(3) rotations。
2. Angular velocity lives in the Lie algebra so(3)，and the equations of motion describe its evolution。
3. Different types of top motion (precession，nutation，free rotation)corresponds to different dynamics behaviors within SO(3)。
4. The conservation laws and symmetries arise from the struture of SO(3)，leading to deep connections with physics and geometry。